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Non-classical symmetries and the singular manifold method: the Burgers and the Burgers–Huxley equations

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Abstract. In this paper a generalization of the direct method of Clarkson and Kruskal for finding similarity reductions of partial differential equations is found and discussed for the Burgers and Burgers–Huxley equations. The generalization incorporates the singular manifold method largely based upon the Painlevé property. This singular manifold can be used as a reduced variable. Furthermore, a sort of inverse procedure is hereby developed through which we find the equations that yield the vector field components associated to the symmetries of the PDE. This procedure also displays the profound relationship among the symmetries and the singular manifold as a reduced variable. The symmetries found in this way are shown to be those corresponding to the so-called non-classical symmetries by Bluman and Cole, and Olver and Rosenau.

1. Introduction

In the wiggly road towards a complete elucidation of the meaning of the word *integrability*, the study and understanding of the similarity reductions (SR) of a given partial differential equation (PDE) has been proved to be both conceptually rewarding and technically successful. In spite of its obvious usefulness, a problem remains unsolved: the relationship (if any) among the different methods used to obtain such SR. This paper is an attempt to clarify this important point in a constructive manner.

There exist two main approaches to the problem of finding similarity reductions of a given partial differential equation:

The Lie approach. Each of the Lie symmetries of a given PDE gives rise to a SR. All that remains is to find the Lie symmetries of the corresponding PDE. In 1974 Bluman and Cole [1] and later Olver and Rosenau [2] generalized the Lie method to encompass symmetry transformations leaving invariant just a subset of the possible solutions of the PDE. These new symmetries have become known as 'conditional symmetries'. We shall be referring to this generalization as the *non-classical method.*

The direct method. The algorithm developed in 1989 by Clarkson and Kruskal [3, 4] which allows us to find SR of the PDE without using group theory at all.

Are these two methods equivalent? For the Boussinesq equation Levi and Winternitz [5] have shown that both methods lead to the same SR. However Clarkson and Nucci [6] and Pucci [7] have argued that in the Fitzhugh–Nagumo and Burgers equation, respectively, the non-classical method yields more information than the direct method since a considerable amount of SR obtained from the former cannot be found using the latter.

More recently the author [8] has proposed a combination of both the direct method and the singular manifold method of Weiss [9] which can be deduced from the generalization of Painlevé analysis for PDE [10]. In this new and consistent framework the present author has been able to prove that one can obtain all the SR already found through the non-classical method for the Fitzhugh–Nagumo equation. Although the specific features of this above mentioned combination will be discussed below we would like to propose at this stage that a consistent use of the direct method should include in the future a systematic use of Painlevé analysis in the way to be described shortly.

This paper deals with both a detailed description of the direct method combined with the singular manifold method and an application to this approach to the Burgers and Burgers–Huxley equations. We shall be able to prove that the same results as those found through the non-classical method [6, 7, 11] can be obtained with the use of our formalism, thus adding more credibility to the conjecture that both approaches yield the same type of information on the SR of a given PDE. The plan of the paper is as follows: in section 2 we analyse the case of the Burgers equation from the point of view of the direct, non-classical and singular manifold method and we show the equivalences and differences we encounter in following these three different roads. Section 3 is entirely devoted to the Burgers–Huxley equation (a case in which only the conditional Painlevé property holds) a similar analysis is performed. Section 4 contains conclusions and prospects for future work.

2. The Burgers equation

We shall start with a case which looks simple but contains some subtleties. We will be analysing first the approach and then the non-classical method which has been proven to yield different information [7]. This apparent mismatch can easily be reconciled by using the singular manifold point of view which gives all information contained in the non-classical method.

2.1. *The direct method for the Burgers equation*

We begin by considering the Burgers equation written in the form

$$u_t+u_{xx}+uu_x=0. \tag{2.1}$$

The direct method of Clarkson and Kruskal [3] aims to find all solutions of (2.1) which can be written as

$$u(x, t)=\alpha(x, t)+z_x\omega[z(x, t)] \tag{2.2}$$

where $z=z(x, t)$ is the reduced variable in such a manner that the PDE (2.1) becomes an ODE for $\omega(z)$.

Substitution of (2.2) into (2.1) leads to

$$\omega_{zz}+\omega\omega_z+(1/z_x^2)(3z_{xx}+z_t+\alpha z_x)\omega_z+(z_{xx}/z_x^2)\omega^2$$
$$+(1/z_x^3)(z_{xxx}+z_{xt}+z_x\alpha_x+z_{xx}\alpha)\omega+(1/z_x^3)(\alpha_t+\alpha_{xx}+\alpha\alpha_x)=0. \tag{2.3}$$

The usual way to proceed in the direct method includes the requirement for (2.3) to be just a second-order ODE for $\omega(z)$ such that all the coefficients in (2.3) are functions of z only. One can easily see [3] that this condition yields the following form for $z(x, t)$ and $\alpha(x, t)$

$$z(x, t) = \theta(t)x + \sigma(t) \tag{2.4a}$$

$$\alpha(x, t) = -(\theta_t/\theta)x - (\sigma_t/\sigma) \tag{2.4b}$$

where $\theta(t)$ and $\sigma(t)$ take the form:

$$\theta(t) = (at^2 + 2bt + e)^{-1/2} \tag{2.5a}$$

$$(ae - b^2)\sigma(t) = -f + \theta[(bc - da)t + (ce - bd)] \tag{2.5b}$$

and a, b, c, d, e and f are arbitrary constants. With these forms for $\theta(t)$ and $\sigma(t)$ the equation (2.3) becomes the following second-order ODE:

$$\omega_{zz} + \omega\omega_z + (ae - b^2)z + f = 0. \tag{2.6}$$

In order to find the symmetries associated to the SR (2.4) one has to obtain the corresponding vector field components. It is well known that these components have to verify the invariant surface condition

$$\xi(x, t, u)u_x + \tau(x, t, u)u_t - \eta(x, t, u) = 0. \tag{2.7}$$

With the help of (2.2) we can write (2.7) in a different form as

$$\xi(\alpha_x + z_{xx}\omega + z_x^2\omega_z) + \tau(\alpha_t + z_{xt}\omega + z_x z_t\omega_z) = \eta. \tag{2.8}$$

Since we need to eliminate all the ω_z dependence from (2.8) one needs to impose the conditions

$$\tau \neq 0 \tag{2.9a}$$

$$\xi/\tau = -z_t/z_x \tag{2.9b}$$

and finally using (2.4) and (2.5) we find for the vector field components the expressions

$$\tau \neq 0 \tag{2.10a}$$

$$\xi/\tau = [(at + b)x + ct + d)]/[at^2 + 2bt + e] \tag{2.10b}$$

$$\eta/\tau = [-(at + b)u + ax + c]/[at^2 + 2bt + e]. \tag{2.10c}$$

A simple comparison with [3] and [7] shows that these correspond to classical Lie symmetries. It is worth noting from (2.9*b*) that this procedure is only suitable for identifying symmetries for which (ξ/τ) are independent of u. This fact has already been noted by other authors [6, 7].

2.2. *The direct method and the singular manifold*

One would be tempted to assume that (2.4) represents the only possible reduction of the general form (2.2). The aim of the present section is to show that this is not the case. In fact (2.2) includes as a particular case the singular manifold method as we shall discuss below. In order to proceed let us first briefly resumé the main features of the application of the singular manifold method to the Burgers equation. More details can be found in [9] and [10]. As is well known, the Burgers equation possesses the Painlevé property [9], or equivalently their solutions can be written in the form:

$$u(x, t)=\sum_{j=0}^{\infty} u_j(x, t)[\phi(x, t)]^{j-1} \tag{2.11}$$

where $u_j(x, t)$ and $\phi(x, t)$ are analytic functions and also $\phi(x, t)=0$ is an arbitrary manifold called the 'movable singularities manifold'. We will assume henceforth that this manifold is non-characteristic ($\phi_x \neq 0$).

To apply the singular manifold method we postulate truncation of the series (2.11) as

$$u=u_1+2(\phi_x/\phi) \tag{2.12}$$

where $u_1(x, t)$ is a solution of (2.1)

$$u_{1t}+u_{1xx}+u_1 u_{1x}=0 \tag{2.13a}$$

and now $\phi(x, t)$ is not an arbitrary function but the 'singular manifold' that satisfies the linear equation

$$\phi_t+\phi_{xx}+u_1\phi_x=0. \tag{2.13b}$$

As is well known [10, 12] the Painlevé property is invariant under homographic transformations. To exhibit this invariance in a more explicit manner it is useful to define the following quantities:

$$v=\phi_{xx}/\phi_x \tag{2.14a}$$

$$s=v_x-(v^2/2) \tag{2.14b}$$

$$w=\phi_t/\phi_x. \tag{2.14c}$$

Notice that both w and s (the Schwartzian derivative) are homographic invariants. These definitions allow us to write (2.13b) in the form:

$$u_1=-v-w \tag{2.15}$$

and substituting this in (2.13a) we are led to

$$w_t=[-2w_x-v_x+(v^2/2)+(w^2/2)]_x \tag{2.16a}$$

where we have used the equation

$$v_t=(w_x+wv)_x \tag{2.16b}$$

that can easily be obtained from (2.14). It is quite trivial to check that the compatibility condition between (2.16a) and (2.16b) leads to the following relation between the homographic invariants:

$$s=v_x-(v^2/2)=-w_x-\tfrac{1}{2}(w+\lambda)^2 \tag{2.17}$$

and also that w must satisfy

$$w_t+w_{xx}-w_x(2w+\lambda)=0 \tag{2.18}$$

where λ is an arbitrary constant.

After this brief review we address ourselves to the main goal of this section by noting that (2.2) contains (2.12) as a particular case. Actually, if we identify in (2.2)

$$z(x,t)=\phi(x,t) \tag{2.19a}$$

$$\alpha(x,t)=u_1(x,t) \tag{2.19b}$$

where ϕ and u_1 satisfy (2.13) the equation (2.3) becomes

$$(2\omega_\phi+\omega^2)_\phi+2(\phi_{xx}/\phi_x^2)(2\omega_\phi+\omega^2)=0. \tag{2.20}$$

Thus (3.9) reduces (3.3) to the first-order ODE:

$$2\omega_\phi+\omega^2=0 \tag{2.21}$$

with obvious solution

$$\omega=2/\phi. \tag{2.22}$$

It is now trivial to see that one can write (2.2) exactly in the form (2.12) as promised.

2.3. *Non-classical symmetries and the singular manifold*

The next step is now to find and identify the symmetries corresponding to use of the singular manifold as a reduced variable. We first observe (appendix 1) that the solution of the form (2.12) verifies

$$u_x=-\tfrac{1}{2}(u-\lambda)(u+\lambda+2w) \tag{2.23a}$$

$$u_t=(u-\lambda)[w_x-(w/2)(u+\lambda+2w)]. \tag{2.23b}$$

Then the surface invariant condition (2.7) is now

$$\eta=(u-\lambda)[\tau w_x-\tfrac{1}{2}(u+\lambda+2w)(\tau w+\xi)]. \tag{2.24}$$

One should keep in mind that still w must satisfy the equation (2.18). We shall be distinguishing henceforth two different cases of (2.24) depending on whether τ is zero or non-zero.

(a) $\tau=0$

In this case we set $\xi=1$ without loss of generality. The vector field components are

$$\tau=0 \tag{2.25a}$$

$$\xi=1 \tag{2.25b}$$

$$\eta=-\tfrac{1}{2}(u-\lambda)(u+\lambda+2w) \tag{2.25c}$$

and from (2.25*c*) we obtain

$$w=-[(u+\lambda)/2]-\eta(u-\lambda)^{-1}. \tag{2.26}$$

Taking into account the fact that w must satisfy (2.18) it is not hard to check that simple substitution of (2.26) into (2.18) leads to the following equation for η:

$$\eta_t+\eta_{xx}+2\eta\eta_{ux}+u\eta_x+\eta^2(1+\eta_{uu})=0. \tag{2.27}$$

Now (2.25*a*), (2.25*b*) and (2.27) exactly correspond to the first of the two non-classical symmetries of Burgers equation found by Pucci [7] and Ames [11].

(b) $\tau \neq 0$

Alternatively we now set $\tau = 1$ without loss of generality. The equation now (2.24) takes the form:

$$\eta = (u-\lambda)[w_x - \tfrac{1}{2}(u+\lambda+2w)(w+\xi)] \tag{2.28}$$

from which one can obtain w_x as

$$w_x = \eta(u-\lambda)^{-1} + \tfrac{1}{2}(u+\lambda+2w)(w+\xi). \tag{2.29}$$

Using (2.18) as in the previous case we find under some tedious but straightforward calculation (see appendix 2):

$$\begin{aligned}(u-\lambda)^{-1}[\eta_t + \eta_{xx} + u\eta_x + 2\eta\xi_x]& \\ +\tfrac{1}{2}(u+\lambda+2w)[\xi_t + \xi_{xx} + (2\xi - u)\xi_x - 2\eta\xi_u - 2\eta_{ux} - \eta]& \\ -\tfrac{1}{2}u_x(u+\lambda+2w)[\eta_{uu} - 2\xi_{ux} + 2(u-\xi)\xi_u]& \\ +\tfrac{1}{2}u_x^2(u+\lambda+2w)\xi_{uu} = 0.&\end{aligned} \tag{2.30}$$

In order for (2.30) to be satisfied independently of w, the vector field components ξ and η must satisfy

$$\xi_{uu} = 0 \tag{2.31a}$$

$$\eta_{uu} - 2\xi_{ux} + 2(u-\xi)\xi_u = 0 \tag{2.31b}$$

$$\xi_t + \xi_{xx} + (2\xi - u)\xi_x - 2\eta\xi_u - 2\eta_{ux} - \eta = 0 \tag{2.31c}$$

$$\eta_t + \eta_{xx} + u\eta_x + 2\eta\xi_x = 0. \tag{2.31d}$$

These equations (2.31) exactly correspond to the second non-classical symmetry found in [7] and [11] through the Bluman and Cole group theoretical procedure [1, 2].

One could even proceed further as to find the specific form of ξ and η. To achieve this goal we note that (2.31a) implies

$$\xi(x,t,u) = \rho(x,t)u + \delta(x,t) \tag{2.32}$$

which, combined with (2.31b) and (2.28), leads to $\rho = -\frac{1}{2}$ so that

$$\tau = 1 \tag{2.33a}$$

$$\xi = -\tfrac{1}{2}u + \delta(x,t) \tag{2.33b}$$

$$\eta = (u-\lambda)\{w_x + \tfrac{1}{2}(u+\lambda+2w)[-(u/2) + w + \delta)]\}. \tag{2.33c}$$

Next we force the above equations to fulfill (2.31c) and (2.31d). Therefore we finally obtain for δ:

$$\delta = -(\lambda/2) - w + (w_x/w) \tag{2.34a}$$

together with the equation (2.18) for w that reads

$$w_t + w_{xx} - w_x(2w+\lambda) = 0. \tag{2.34b}$$

Therefore we have shown that any solution of the form (2.12) possesses a non-classical symmetry whose vector field components can be calculated through (2.33) where w and

δ satisfy (2.34*a*, *b*). Besides this, system (2.34) gives a systematic procedure for generating solutions of Burgers equation. For each solution of (2.34) one can solve the Ricatti equation (2.17) for v. With the help of (2.14) and (2.15) one can also find $\phi(x, t)$ and $u_1(x, t)$. Inserting ϕ and u_1 in (2.12) one obtains a solution of (2.1). In appendix 3 it is shown that the solutions (iii_a), (iii_b) and (iii_c) of [7] are just particular cases of (2.34) obtained for δ = constant.

3. The Burgers–Huxley equation

We now want to show that the singular manifold method is strong enough to hold even in the absence of the Painlevé property. The celebrated Fitzhugh–Nagumo equation [6] constitutes such an example and the author [8] has already used the singular manifold analysis to analyse the fact that the direct method was unable to reproduce the results provided by the non-classical method. The Fitzhugh–Nagumo equation does not possess the Painlevé property but it has the conditional Painlevé property instead, which refers to the fact that the Painlevé property is enjoyed just by a subset of the solutions of the PDE under study. Besides, we shall generalize the results of [8] by looking at the generalization of the Fitzhugh–Nagumo equation called the Burgers–Huxley equation from which the former is a particular case of the latter. The Burgers–Huxley equation is

$$u_t - u_{xx} + \alpha u u_x + \beta u(u-1)(u-\gamma) = 0. \tag{3.1}$$

The Fitzhugh–Nagumo equation corresponds to the choice of parameters $\alpha = 0$; $\beta = 1$.

3.1. The singular manifold method

Let us apply to (3.1) the singular manifold method. This has been applied by the author and collaborators [13] to obtain particular solutions of this equation. Here we are interested in another aspect of the analysis. In agreement with [13] one can obtain the truncated solutions as

$$u = u_1 - \lambda(\phi_x/\phi) \tag{3.2}$$

where λ is a parameter satisfying

$$\beta\lambda^2 + \alpha\lambda - 2 = 0 \tag{3.3}$$

and u_1 must be expressed in terms of the quantities (2.14) as

$$u_1 = \tfrac{1}{2}[v - qw + a(1-q)]. \tag{3.4}$$

The constants a and q are defined as

$$q = (3 - \alpha\lambda)^{-1} \tag{3.5a}$$

$$a = (1+\gamma)/\lambda. \tag{3.5b}$$

The equation under study has been shown not to possess the Painlevé property [13]. However, we can always use (3.2) to generate solutions. In particular one can always start with the trivial solution $u_1 = 0$ as is done, for instance, in [13]. Thus if we set $u_1 = 0$ in (3.4) one can express v in terms of w in the form:

$$v = qw + a(q-1) \tag{3.6}$$

and since ($u_1=0$) from (3.2)

$$u=-\lambda(\phi_x/\phi) \tag{3.7}$$

the solution u must obviously verify (3.1). Imposing this condition and using (3.6) one obtains a pair of equations on w that define the singular manifold, namely:

$$qw_x+(qw+c_1)(qw+c_2)=0 \tag{3.8a}$$

$$qw_t=-aw_x. \tag{3.8b}$$

The constants c_1 and c_2 are defined as

$$c_1=qa-(\gamma/\lambda) \tag{3.9a}$$

$$c_2=qa-(1/\lambda). \tag{3.9b}$$

3.2. *Non-classical symmetries and the singular manifold*

In order to obtain u_x and u_t we take the derivatives of (3.7) and then use (3.7) itself to eliminate (ϕ_x/ϕ). The procedure is exactly the same as used in appendix 1 for the Burgers equation and the result is

$$u_x=u[qw+a(q-1)+(u/\lambda)] \tag{3.10a}$$

$$u_t=u\{w[(u/\lambda)-qa]-(c_1c_2/q)\}. \tag{3.10b}$$

Now we turn our attention to the relationship with the non-classical method. The surface invariant condition (2.8)

$$\xi(x,t,u)u_x+\tau(x,t,u)u_t=\eta(x,t,u) \tag{3.11}$$

with u_x and u_t given by (3.10), is such that w must satisfy (3.8). Let us consider as usual the cases $\tau=0$ and $\tau\neq0$, separately.

(a) $\tau=0$

We set $\xi=1$ without loss of generality such that the surface invariance condition now reads

$$\eta=u_x=u[qw+a(q-1)+(u/\lambda)] \tag{3.12}$$

where w satisfies (3.8). To see that (3.12) corresponds to a non-classical symmetry can be seen (appendix 4) by taking w from (3.12) and inserting its form in (3.8). The resulting equations for η are

$$\eta_x+\eta\eta_u-\eta[(3u/\lambda)-a]+(1/\lambda^2)u(u-1)(u-\gamma)=0 \tag{3.13a}$$

$$\eta_t+\{\eta[(3u/\lambda)-a-\alpha u]-(\beta+\lambda^{-2})u(u-1)(u-\gamma)\}\{\eta_u-(\eta/u)-(u/\lambda)\}$$
$$-a\{(\eta^2/u)-(\eta/\lambda)(2u-a\lambda)+\lambda^{-2}u(u-1)(u-\gamma)\}=0. \tag{3.13b}$$

These equations (3.13a, b) satisfy the equation (appendix 4) which one usually obtains through the non-classical method for the $\tau=0$ case. This equation is

$$\eta_t-\eta_{xx}-\eta^2\eta_{uu}-2\eta\eta_{ux}-\beta u(u-1)(u-\gamma)\eta_u$$
$$+\beta\eta[3u^2-2(\gamma+1)u+\gamma]+\alpha(\eta^2+u\eta_x)=0. \tag{3.14}$$

Therefore the truncated solutions obtained in [13] have a non-classical symmetry given by (3.12). We also obtain through the SMM the solution of the rather complicated PDE (3.14).

(b) $\tau \neq 0$

Now we set $\tau = 1$. Substituting (3.10) into (3.11) one obtains the following relationship between η and ξ:

$$\eta = u\{w[(u/\lambda) - qa] - (c_1 c_2/q)\} + \xi u[qw + a(q-1) + (u/\lambda)]. \tag{3.15}$$

As in the procedure developed for the Burgers equation satisfy the symmetry equations can be obtained by taking w from (3.10*a*) and forcing this form of w to satisfy (3.8). The result (appendix 4) yields two different types of symmetry.

(b.1)

$$\xi = a - (u/\lambda q) \tag{3.16a}$$

$$\eta = -(\lambda^2 q)^{-1} u(u-1)(u-\gamma) \tag{3.16b}$$

which do not require any additional condition for w. Thus, all solutions obtained in [13] possess this symmetry without further restriction.

(b.2)

$$\xi = -w \tag{3.17a}$$

$$\eta = 0. \tag{3.17b}$$

In this case w must verify

$$(qw + c_1)(qw + c_2) = 0. \tag{3.17c}$$

If one wishes to check that b.1, 2 are non-classical symmetries it would be enough to write the symmetry equations obtained through the non-classical method by Bluman and Cole ($\tau \neq 0$) which are

$$\xi_{uu} = 0 \tag{3.18a}$$

$$\eta_{uu} - 2\xi_{xu} - 2\xi_u(\alpha u - \xi) = 0 \tag{3.18b}$$

$$\eta_t - \eta_{xx} + 2[\eta + \beta u(u-1)(u-\gamma)]\xi_x + \alpha u \eta_x$$
$$+ \beta \eta[3u^2 - 2(\gamma+1)u + \gamma] - \beta u(u-1)(u-\gamma)\eta_u = 0 \tag{3.18c}$$

$$\xi_t - \xi_{xx} - [\alpha u - 2\xi]\xi_x - \xi_u[2\eta + 3\beta u(u-1)(u-g)] - \alpha\eta + 2\eta_{ux} = 0 \tag{3.18d}$$

and to find the solution, which in this case is quite simple. This solution leads directly to the symmetries given in (3.16) and (3.17). Note that the symmetries b.1 for the Fitzhugh–Nagumo equation are the aims of the discussion contained in [6] since, as we shall see in a moment, this cannot be obtained through the direct method.

3.3. The direct method and the singular manifold

In fact, in order to establish the connection between the singular manifold method and the direct method the singular manifold ϕ is supposed to be the reduced variable $z(x, t)$.

The solution (3.7) can be expressed now as:

For the direct method the solution of (3.1) is supposed to be

$$u=z_x\omega(\phi) \tag{3.19}$$

where $\omega(\phi)$ must satisfy the equation

$$\omega_{zz}+\alpha\omega\omega_z+\omega^3+(\omega^2/z_x^2)[-\beta z_x(1+\gamma)+\alpha z_{xx}] \\ +(\omega_z/z_x^2)[z_t-3z_{xx}]+\omega[z_{xt}-z_{xxx}+\gamma\beta z_x]=0.$$

The direct method requires (3.20) as a second-order ODE. The only possibility for this is

$$z_t/z_x=-c_i/q \tag{3.21}$$

where q and c_i are defined in (3.5) and (3.9). This means that the only possible reduction that can be identified using the direct method corresponds to the travelling wave reduction. That is to say

$$z=[x-(ct/q)+x_0] \tag{3.22}$$

and the associated symmetry is obviously (b.2).

On the other hand the connection between the direct method and the singular manifold method can be established if we identify z in (3.20) as the singular manifold ϕ. In this case, and taking into account the equations (3.6) and (3.8), the equation (3.20) reduces to the first-order ODE:

$$\lambda\omega(\phi)-\omega^2=0 \tag{3.23}$$

whose obvious solution is

$$\omega=-\lambda/\phi \tag{3.24}$$

which, combined with (3.19), is exactly the truncation ansatz (3.7) for the singular manifold method.

4. Conclusions

In this paper we have dealt with the problem of the symmetries of the Burgers and the Burgers–Huxley equations. The similarity reductions giving rise to these symmetries can indeed be found using group theoretical methods and can be classified in two different categories: classical (conventional Lie theory) and non-classical (Bluman and Cole, and Olver and Rosenau).

Can one reproduce these two classes of symmetries through direct methods? Some authors have been rather reluctant to believe that the direct methods were in any way suitable for describing both different classes of symmetries. What we have shown in this paper is that the singular manifold method is indeed able to reproduce the results already obtained using group theory. In this paper we have shown this for two equations, but several other examples already worked out [8] confirm our expectations, which can be summarized in the following way: the direct method of Clarkson and Kruskal should be extended to include the results from the singular manifold analysis.

This new *improved direct method or singular manifold method* is equivalent to the non-classical method in all examples we know of. A proof of the total equivalence is still lacking but work in this direction is now in progress.

An interesting remark allows one to classify the symmetries of the two different classes from our new point of view: with the direct method of Clarkson and Kruskal one is able to obtain SR transforming (2.1) in a second-order ODE. If we now use the singular manifold method the SR which reduce (2.1) to a first-order ODE are obtained. Actually the singular manifold plays the role of the reduced variable itself. We can also use a procedure to obtain the vector field components associated to the use of the singular manifold as a reduced variable. The interesting fact is that these symmetries are exactly the ones obtained through the Bluman and Cole non-classical method.

It is also interesting to emphasize that the Painlevé property is by no means a necessary condition for successfully applying the singular manifold procedure. In fact the conditional Painlevé property seems the only necessary requirement for applying this method successfully.

If our conjecture is true, a crucial question remains still unanswered. This question concerns the internal relationships between symmetries and the singular manifold method, which lies at the heart of a complete understanding of the Painlevé analysis. We leave these as another related question for future work, to be reported elsewhere.

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Appendix 1

In this appendix we aim to find u_x and u_t when u is defined by (2.12). We take the derivative of (2.12) with respect to x

$$u_x = u_{1x} + 2v(\phi_x/\phi) - 2(\phi_x/\phi)^2 \tag{A1.1}$$

where v is defined in (2.14*a*). Eliminating ϕ_x/ϕ between (2.12) and (A1.1)

$$u_x = u_{1x} + v(u - u_1) - \tfrac{1}{2}(u - u_1)^2. \tag{A1.2}$$

Now we use (2.15):

$$u_x = -v_x - w_x + \tfrac{1}{2}(v^2 - w^2) - wu - \tfrac{1}{2}u^2 \tag{A1.3}$$

and with the aid of (2.17) finally we have

$$u_x = -\tfrac{1}{2}(u - \lambda)(u + \lambda + 2w). \tag{A1.4}$$

This is just the expression used in (2.23*a*).

In the same way differentiating (2.12) with respect to t

$$u_t = u_{1t} + 2(w_x + wv)(\phi_x/\phi) - 2w(\phi_x/\phi)^2 \tag{A1.5}$$

and again we can use (2.12) in order to eliminate ϕ_x/ϕ:

$$u_t=u_{1t}-u_1(w_x+wv)-\tfrac{1}{2}wu_1^2+u(w_x+wv+u_1w)-\tfrac{1}{2}wu^2. \tag{A1.6}$$

Using (2.17) we obtain

$$u_t=v_t-w_t+w_x(w+v)+\tfrac{1}{2}w(v^2-w^2)+u(w_x-w^2)-\tfrac{1}{2}wu^2 \tag{A1.7}$$

and with the aid of (2.16*a*), (2.16*b*) and (2.17)

$$u_t=(u-\lambda)[w_x-(w/2)(u+\lambda+2w)]. \tag{A1.8}$$

This is the expression used in (2.23*b*).

Appendix 2

Our purpose in this appendix is to obtain from (2.29) the equations (2.31) for the vector field components η and ξ using also the fact that w must satisfy (2.18). We differentiate (2.29) with respect to t and (2.18) with respect to x, obtaining

$$\begin{aligned}w_{xt}=&-(u-\lambda)^{-2}\eta u_t+(u-\lambda)(\eta_t+u_t\eta_u)+\tfrac{1}{2}(u_t+2w_t)(w+\xi)\\&+\tfrac{1}{2}(u+\lambda+2w)(w_t+\xi_t+u_t\xi_u)\end{aligned} \tag{A2.1}$$

$$w_{xt}+(w_x-w^2-\lambda w)_{xx}=0. \tag{A2.2}$$

The combination of (A2.1) and (A2.2) yields

$$\begin{aligned}&2(u-\lambda)^{-3}u_x^2\eta-2(u-\lambda)^{-2}u_x(\eta_x+u_x\eta_u)-(u-\lambda)^{-2}\eta(u_t+u_{xx})\\&+(u-\lambda)[\eta_t+\eta_{xx}+\eta_u(u_t+u_{xx})+2u_x\eta_{ux}+u_x^2\eta_{uu}]\\&+(\xi/2)(u_t+u_{xx}+2w_t+2w_{xx})+(w/2)(u_t+u_{xx})+(w_t/2)(4w+u+\lambda)\\&+\tfrac{1}{2}(u+\lambda+2w)[\xi_t+\xi_{xx}+\xi_u(u_t+u_{xx})+2u_x\xi_{ux}+u_x^2\xi_{uu}]\\&+(\xi_x+u_x\xi_u)(u_x+2w_x)+(w_{xx}/2)(u-\lambda)+w_xu_x=0.\end{aligned} \tag{A2.3}$$

Using (2.1) and (2.18)

$$\begin{aligned}&(u-\lambda)^{-1}\{2(u-\lambda)^{-2}u_x^2\eta+[u\eta-2\eta_x+\eta(2w+\lambda)](u-\lambda)^{-1}u_x\\&+\eta_t+\eta_{xx}+2\eta\xi_x-\eta_x(2w+\lambda)\}\\&+\{-2\eta_u(u-\lambda)^{-2}u_x^2+[2\eta_{ux}-(u+2w+\lambda)\eta_u+2\eta\xi_u+\eta](u-\lambda)^{-1}u_x\}\\&+\{\eta_{uu}(u-\lambda)^{-1}u_x^2+(u+2w+\lambda)u_x[\xi_u(\xi-u)+\xi_{ux}]\}\\&+\tfrac{1}{2}(u+2w+\lambda)[\xi_x(2\xi-u)+\xi_t+\xi_{xx}+u_x^2\xi_{uu}]=0\end{aligned} \tag{A2.4}$$

and with the aid of (2.23*a*)

$$\begin{aligned}&(u-\lambda)^{-1}[\eta_t+\eta_{xx}+u\eta_x+2\eta\xi_x]\\&+\tfrac{1}{2}(u+\lambda+2w)[\xi_t+\xi_{xx}+(2\xi-u)\xi_x-2\eta\xi_u-2\eta_{ux}-\eta]\\&-\tfrac{1}{2}u_x(u+\lambda+2w)[\eta_{uu}-2\xi_{ux}+2(u-\xi)\xi_u]\\&+\tfrac{1}{2}u_x^2(u+\lambda+2w)\xi_{uu}=0.\end{aligned} \tag{A2.5}$$

This is the equation (2.30) which we were interested in.

Appendix 3

The simplest case of (2.34) corresponds to δ = constant. In this case the equations (2.34) are

$$w_x - w^2 = \tfrac{1}{2}(2\delta + \lambda)w \tag{A3.1}$$

$$w_t - \tfrac{1}{2}(\lambda - 2\delta)w_x = 0. \tag{A3.2}$$

On the other hand, choosing $u_1 = \lambda$ in (2.15), we have

$$v = -(w + \lambda). \tag{A3.3}$$

This is obviously a solution of the Ricatti equation (2.17).

It is easy to verify that the solution of (A3.1) and (A3.2) is

$$w = -v - \lambda = -k\{1 + \tanh[k(x + ct + x_0)]\} \tag{A3.4}$$

with $k = \frac{1}{4}(\lambda + 2\delta)$ and $c = \lambda - 2k$ or

$$w = -v = -2C(2Cx + B)^{-1} \qquad \text{if} \qquad \delta = \lambda = 0 \tag{A3.5}$$

where x_0, B and C are integration constants.

We can now use the equations (2.14) in order to obtain $\phi(x, t)$. The result is:

(a) $\qquad \phi = A + Bx + C(x^2 - 2t) \qquad (\text{for } \lambda = \delta = 0)$

and the solution (2.12) is in this case

$$u = 2(B + 2Cx)(A + Bx + Cx^2 - 2Ct)^{-1} \tag{A3.6}$$

i.e. the solution (iii_a) of [7].

(b) $\qquad \phi = A + B\exp[-(\lambda/2)x + (\lambda^2/4)t] + C\exp(-\lambda x) \qquad (\text{for } \delta = 0, \lambda \neq 0).$

Using this expression for ϕ in (2.12) we have the following expression for u:

$$u = \lambda[A - C\exp(-\lambda x)]\{A + B\exp[-(\lambda/2)x + (\lambda^2/4)t] + C\exp(-\lambda x)\}^{-1}. \tag{A3.7}$$

This solution corresponds to (iii_b) in [7].

(c) $\qquad \phi = A + B\exp(\delta x - \delta^2 t) + Cx \qquad (\text{for } \delta \neq 0, \lambda = 0).$

Now the solution (2.12) is

$$u = 2[\delta B\exp(\delta x - \delta^2 t) + C][A + B\exp(\delta x - \delta^2 t) + Cx]^{-1} \tag{A3.8}$$

and this is the solution (iii_c) of [7].

Appendix 4

In order to determine the equations for the non-classical symmetries of the Burgers–Huxley equation we must distinguish between the $\tau = 0$ and $\tau \neq 0$ cases.

(a) $\tau = 0$

We need to use (3.12) in the form:

$$qw = (\eta/u) - (u/\lambda) - a(q - 1). \tag{A4.1}$$

Inserting this in (3.8*a*):

$$D_x\eta-\eta(u_x/u)-(uu_x/\lambda)+u^{-1}[\eta-u(u-1)/\lambda][\eta-u(u-\gamma)/\lambda]. \quad \text{(A4.2)}$$

Using partial derivatives and

$$u_x=\eta(u,x,t) \quad \text{(A4.3)}$$

the resulting equation is

$$\eta_x+\eta\eta_u-\eta[(3u/\lambda)-a]+\lambda^{-2}u(u-1)(u-\gamma). \quad \text{(A4.4)}$$

On the other hand the substitution of (A3.1) and (A3.2) in (3.8*b*) yields

$$D_t\eta-\eta(u_t/u)-(uu_t/\lambda)-au^{-1}[\eta-u(u-1)/\lambda][\eta-u(u-\gamma)/\lambda]=0. \quad \text{(A4.5)}$$

Now we can combine the derivative of (A4.2) with (A4.5) in order to obtain the non-classical symmetry equation (3.14). This is to say

$$\eta_t-\eta_{xx}-\eta^2\eta_{uu}-2\eta\eta_{ux}-\beta u(u-1)(u-\gamma)\eta_u$$
$$+\beta\eta[3u^2-2(\gamma+1)u+\gamma]+\alpha(\eta^2+u\eta_x)=0 \quad \text{(A4.6)}$$

where we have used (3.1) in the form

$$u_t=D_x\eta-\alpha u\eta-\beta u(u-1)(u-\gamma). \quad \text{(A4.7)}$$

(b) $\tau\neq 0$

Now from (6.10*a*)

$$qw=(u_x/u)-a(q-1)-(u/\lambda) \quad \text{(A4.8)}$$

so that

$$qw_x=(A/u)+\alpha u_x+\beta(u-1)(u-\gamma)-(u_x/\lambda)-(u_x/u)^2 \quad \text{(A4.9)}$$

where, as always

$$A=\eta-\xi u_x=u_t. \quad \text{(A4.10)}$$

Using (A4.8, 9) in (3.8*a*) the result is

$$u_x[-\xi+a-(u/q\lambda)]+[\eta+(\lambda^2q)^{-1}u(u-1)(u-\gamma)]=0. \quad \text{(A4.11)}$$

There are two possible solutions of (A4.11) depending on whether w is or is not a constant solution of (3.8).

(b.1) If w is an arbitrary solution of (3.8), (A4.11) is satisfied only if

$$\xi=a-(u/q\lambda) \quad \text{(A4.12}a\text{)}$$

and

$$\eta=-(\lambda^2q)^{-1}u(u-1)(u-\gamma). \quad \text{(A4.12}b\text{)}$$

(b.2) On the other hand if w is a constant solution of (3.8*a*) (namely $qw=-c_i$ ($i=1,2$)) there is another possibility because in this case (A4.8) means

$$u_x=u(u-\delta_i)/\lambda \qquad \text{for} \qquad \delta_1=1,\ \delta_2=\gamma \quad \text{(A4.13)}$$

so that, using (A4.13) in (A4.11), the solution is

$$\xi = c_i/q = -w \tag{A4.14a}$$

$$\eta = 0. \tag{A4.14b}$$

References

[1] Bluman G W and Cole J D 1974 *Similarity Methods for Differential Equations* (Springer)
[2] Olver P and Rosenau P 1986 *Phys. Lett.* **114A** 107
[3] Clarkson P A and Kruskal M D 1989 *J. Math. Phys.* **30** 2201
[4] Clarkson P A 1989 *J. Phys. A: Math. Gen.* **22** 3821
[5] Levi D and Winternitz P 1991 *Physica* 49 257
[6] Nucci M C and Clarkson P A 1992 *Phys. Lett.* **164A** 49
[7] Pucci E 1992 *J. Phys. A: Math. Gen.* **25** 2631
[8] Estévez P G 1992 *Phys. Lett.* **171A** 259
[9] Weiss J 1983 *J. Math. Phys.* **24** 1405
[10] Weiss J, Tabor M and Carnevale G 1983 *J. Math. Phys.* **24** 522
[11] Ames W F 1972 *Non Linear Partial Differential Equations in Engineering* (New York: Academic)
[12] Mussete M and Conte R 1991 *J. Math. Phys.* **32** 1450
[13] Estévez P G and Gordoa P R 1990 *J. Phys. A: Math. Gen.* **23** 4831